Uniform Continuity

Mathematics

One delta for the whole domain, and why compactness gives it free.

f is uniformly continuous when a single δ works at every point at once, not a different δ for each point.

$$\forall \varepsilon\ \exists \delta\ \forall x, y : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

$f(x) = x$ on $\mathbb{R}$ — **Uniformly continuous**

Take $\delta = \varepsilon$.

$f(x) = x^2$ on $\mathbb{R}$ — **Not uniformly**

Slope grows without end.

$f(x) = x^2$ on $[0, 1]$ — **Uniformly continuous**

A compact domain.

THE COMMON MISTAKE

× **Assuming continuous means uniformly continuous**

✓ **It does on a compact set, not in general**

$1/x$ is continuous on $(0, 1)$ and not uniformly so. Compactness of the domain is what upgrades the one to the other.

Every Lipschitz function is uniformly continuous. The converse fails: $\sqrt{x}$ on $[0, 1]$ is uniformly continuous and not Lipschitz.