Existence & Uniqueness

Mathematics

What guarantees a solution, what guarantees only one, and how local both are.

If f is continuous near the initial point a solution exists; if the partial of f in y is also continuous, it is unique.

f continuous ⇒ exists · ∂f/∂y too ⇒ unique

$y' = y, y(0) = 1$ — **Unique: y = e^x**

Both conditions hold.

$y' = \sqrt{y}, y(0) = 0$ — **Not unique**

∂f/∂y blows up.

$y' = 1 + y^2, y(0) = 0$ — **Unique but escapes**

Only local existence.

THE COMMON MISTAKE

× Reading the theorem as a global promise

✓ It is local: some interval around x_0

$y' = 1 + y^2$ with $y(0) = 0$ gives $y = \tan x$, which exists only up to $\pi/2$ even though f is perfectly smooth everywhere.

Lipschitz continuity in y is the sharp condition. Continuity of the partial derivative is the version that is easy to check.